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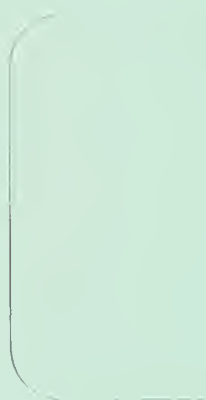
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A NOTE ON COOPERATIVE GAMES
WITH VARYING POWER

Alvin E. Roth

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College of Commerce and Business Administration
University of Illinois at Urbana - Champaign



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A Note on Cooperative Games with Varying Power

by

Alvin E. Roth

The purpose of this note is to consider some properties of cooperative games in which the power of coalitions is dependent on their wealth. In particular, we will want to investigate repeated play of such games, in which the outcome of earlier plays of the game determines the wealth, and thus the power, of coalitions in subsequent plays of the game.

Non-cooperative games in which power is dependent on previous plays of the game have been introduced by Oskar Morgenstern (1972), and called power games. We shall use the same name in our discussion, it being understood that we speak here of cooperative games.

Games in which power depends on wealth arise in a natural way - ordinary market games are of this type. In general, by a market game (with exchange and production) we will mean a set of players $N = \{1, \dots, n\}$, with a set of initial (vector) endowments $\omega^1, \dots, \omega^n$, and a common production set Y . An outcome of the game is some distribution of wealth $x = (x^1, \dots, x^n)$ such that $\sum x^i - \sum \omega^i \in Y$. The characteristic function of each coalition $S \subseteq N$ is customarily taken to be $V(S) = \{(x^1, \dots, x^n) \mid \sum_{i \in S} x^i - \sum_{i \in S} \omega^i \in Y\}$

This is normally interpreted to mean that each coalition has the power to dispose of its initial endowment in any way it pleases. We say that a coalition S is effective for each allocation x in $V(S)$.

The description of the game is completed by a preference ordering for each player on the set of outcomes. If an outcome $x \in V(S)$ is preferred to an outcome y by all members of some coalition, $S \subseteq N$ then we say x dominates y . The set of undominated outcomes is defined to be the core of the game.

It is clear that the characteristics function, and consequently the core, are functions of the initial endowment ω . We shall denote the market resulting from the endowment ω by $M(\omega)$, and the core by $C(\omega)$.

If a game is to be played only once, every outcome in the core can be considered stable, since no coalition of players can redistribute their initial allocation in a way which they unanimously prefer to a given core allocation. However

if a game is to be played more than once, the initial endowment of a given play is the allocation which results from the preceding play.

Thus in allocation x resulting from a market $M(\omega)$ gives rise to a new market $M(x)$. Therefore an allocation x in the core of the market $M(\omega)$ can only be considered stable if x is in the core of the market $M(x)$. The following example shows that not all points in the core of a market game are necessarily stable.

Example: The game consists of two players $N=\{1,2\}$, and three commodities. An allocation of goods to player i is denoted by $x^i = (x_1^i, x_2^i, x_3^i)$, where superscripts index players and subscripts index commodities.

The initial allocations of the players are $\omega^1 = (1,0,0)$ and $\omega^2 = (0,1,0)$.

The production set permits any amount of the first commodity to be transformed into an equal amount of the third commodity: i.e. the production set is

$Y = \{ (x^1, x^2) \mid \sum_{i=1,2} x_2^i = 0 \quad \sum_{i=1,2} (x_2^i + x_3^i) = 0 \}$. The preferences of the two players for outcomes $x = (x^1, x^2)$ are represented by the utility functions

$u_1(x) = x_2^1 - 2x_3^2$ and $u_2(x) = x_1^2 + 2x_3^2$. Thus the initial endowment ω gives each player a utility of zero.

The characteristic function of the market game $M(\omega)$ is $V(1) = \{x \mid x^1 \leq \omega^1\}$; $V(2) = \{x \mid x^2 \leq \omega^2\}$; $V(12) = \{x \mid (x, 0) \in Y\}$. The core of the game $M(\omega)$ contains the allocation $x = [x^1, x^2] = [(0,1,0); (1,0,0)]$ such that $u_1(x) = u_2(x) = 1$.

In particular, the allocation $y = [(0,1,0); (0,0,1)]$ with $u_1(y) = -1$ and $u_2(y) = 2$ does not dominate x in the game $M(\omega)$, since player 2, who prefers y to x , is not effective for y . However in the game $M(x)$ player 2 is effective for the allocation y , and thus x is not in the core of the game $M(x)$. The only stable allocation in this example is the allocation y .

The example can perhaps be illuminated by a simple 'story'. The first two commodities are private goods; uranium and money. The third commodity, which can be produced from the first, is a public good which gives utility to player 2 and disutility to player 1 - atomic weapons. Player 1 initially holds all the uranium and player 2 holds all of the money. The allocation x

represents the mutually profitable trade of uranium for money. The observation that x is unstable simply reflects the fact that once player 2 is in possession of the uranium, he has the incentive to divert it to weapons production to the detriment of player 1. Thus, the allocation x can only be considered stable if the rules of the game provide for an enforceable agreement limiting the future use to which uranium can be put. Under the more usual (and realistic) assumption that agreements can only be enforced when the commodities in question remain under the physical control of the parties involved, we see that player 2 has both the power and the incentive to reallocate the uranium for weapons production.¹

When utility is freely transferable it is often convenient to consider games with side payments. We define a power game with side payments to be a collection (N, X, x^0, V_X) , where $N = \{1, \dots, n\}$ is the set of players, X is the convex set of outcomes (utility n -tuples), x^0 is the initial outcome, and $V_X = \{v_x | x \in X\}$ is a set of real-valued characteristic functions v_x corresponding to each outcome $x \in X$. Alternatively, view V_X as a function $V_X: X \rightarrow \mathbb{R}^N$. Thus the power of each coalition, which is reflected by the characteristic function, is dependent on the utility of each player.

The core of a characteristic function v_x is the set $C(x) = \{x \in X | \text{for all } S \subseteq N, \sum_{i \in S} x_i \geq v_x(S)\}$. An outcome $x \in X$ is stable if and only if $x \in C(x)$. A characteristic function v_x with a non-empty core $C(x)$ is said to be balanced. The function V_X is said to be balanced if $C(x)$ is non-empty for each $x \in X$.

Example 2: Consider the power game (N, X, x^0, V_X) in which $N = \{1, 2\}$, $X = \{(x_1, x_2) | x_1 + x_2 \leq 1; x_1, x_2 \geq 0\}$, $x^0 = (0, 0)$, and for each $x \in X$

$$v_x(1) = \min [1, \max (0, 2x_1 - x_2)]$$

$$v_x(2) = \min [1, \max (0, 2x_2 - x_1)]$$

$$v_x(12) = 1$$

Note that if $x_i = x_j$ then $v_x(i) = x_i$ and $v_x(j) = x_j$ for $i=1, 2$ while if $x_i > x_j$ then

¹ Similar phenomena have been observed by Rosenthal (1972) and Roth & Postlewaite (1

$$v_x(i) > x_i \text{ and } v_x(j) < x_j.$$

The dynamic behavior of this game is aptly expressed by the proverb "the rich get richer and the poor get poorer." In particular, it can readily be verified that the only stable outcomes of this game are $(\frac{1}{2}, \frac{1}{2})$, $(1, 0)$, and $(0, 1)$.

In general, we can prove the following:

Theorem: For every power game with side payments in which V_X is balanced and a continuous function of outcomes, the set of stable allocations is non-empty.

Proof: For every outcome $x \in X$, the correspondence $C(x)$ is a non-empty, closed, convex polyhedron. We want to show that $C(x)$ is upper-semi continuous: i.e., if $x^{(1)} \in X$ and $\lim_{i \rightarrow \infty} x^{(i)} = x$, and if $y^{(1)} \in C(x^{(1)})$ and $\lim_{i \rightarrow \infty} y^{(i)} = y$ then $y \in C(x)$.

Suppose not. Then there exists a coalition $S \subseteq N$ such that $v_x(S) > \sum_{i \in S} y^i$. By continuity, there must be an N'' such that for all $n > N$, $v_{x^{(n)}}(S) > \sum_{i \in S} (y^{(n)})^i$ which contradicts the fact that $y^{(n)} \in C(x^{(n)})$. So $C(x)$ is upper semi-continuous.

Hence by Kakutani's fixed point theorem [1941] there exists an outcome x such that $x \in C(x^{(n)})$: i.e. there exists a stable outcome.

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